Problem 2.9

Suppose the electric field in some region is found to be $\mathbf{E} = kr^3\hat{\mathbf{r}}$, in spherical coordinates (k is some constant).

(a) Find the charge density ρ.

(b) Find the total charge contained in a sphere of radius R, centered at the origin. (Do it two different ways.)

Solution

One of the governing equations for the electric field in vacuum is Gauss's law.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Solve for ρ and expand the divergence operator in spherical coordinates (r, ϕ, θ), where θ is the angle from the polar axis.

$$\begin{aligned}
\rho &= \epsilon_0 \nabla \cdot \mathbf{E} \\
&= \epsilon_0 \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(E_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}\right] \\
&= \epsilon_0 \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \cdot kr^3) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(0 \cdot \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(0)\right] \\
&= \epsilon_0 \left[\frac{1}{r^2}\frac{d}{dr}(kr^5)\right] \\
&= \epsilon_0 \left[\frac{1}{r^2}(5kr^4)\right] \\
&= 5\epsilon_0 k r^2
\end{aligned}$$

Integrate the charge density over the volume of a sphere of radius R to find the charge enclosed.

$$\begin{aligned}
Q_{\text{enclosed}} &= \iiint\limits_{x^2+y^2+z^2 \le R^2} \rho \, dV \\
&= \int_0^\pi \int_0^{2\pi} \int_0^R (5\epsilon_0 k r^2)(r^2 \sin\theta \, dr \, d\phi \, d\theta) \\
&= 5\epsilon_0 k \left(\int_0^\pi \sin\theta \, d\theta\right)\left(\int_0^{2\pi} d\phi\right)\left(\int_0^R r^4 \, dr\right) \\
&= 5\epsilon_0 k(2)(2\pi)\left(\frac{R^5}{5}\right) \\
&= 4\pi\epsilon_0 k R^5
\end{aligned}$$

Alternatively, use Gauss's law and the divergence theorem to find the charge enclosed.

$$
\begin{aligned}
Q_{\text{enclosed}} &= \iiint\limits_{x^2+y^2+z^2\leq R^2} \rho \, dV \\
&= \iiint\limits_{x^2+y^2+z^2\leq R^2} (\epsilon_0 \nabla \cdot \mathbf{E}) \, dV \\
&= \epsilon_0 \iiint\limits_{x^2+y^2+z^2\leq R^2} \nabla \cdot \mathbf{E} \, dV \\
&= \epsilon_0 \oiint\limits_{x^2+y^2+z^2=R^2} \mathbf{E} \cdot d\mathbf{S} \\
&= \epsilon_0 \oiint\limits_{x^2+y^2+z^2=R^2} (kR^3 \hat{\mathbf{r}}) \cdot (\hat{\mathbf{r}} \, dS) \qquad \text{(On the surface, } \mathbf{E} = kR^3\hat{\mathbf{r}}.) \\
&= \epsilon_0 \oiint\limits_{x^2+y^2+z^2=R^2} (kR^3) \, dS \\
&= \epsilon_0 k R^3 \oiint\limits_{x^2+y^2+z^2=R^2} dS \\
&= \epsilon_0 k R^3 (4\pi R^2) \\
&= 4\pi\epsilon_0 k R^5
\end{aligned}
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